A Note on Covariance Function of a Regime Switching AR (1) Process

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Abstract

This paper is concerned with the auto-covariance function (ACVF) of a regime switching AR (1) process. In this model, two independent Markov chains govern on auto-regressive coefficient and standard deviation of white noise process. Our approach to solve this problem is to obtain the ACVF of a AR(1) model with time varying parameters and then to extend this result to regime switching case. An application of our formulae in model selection is proposed. Finally, a conclusion section is also given.

Keywords

Auto-covariance Function; Auto-regressive Model; Markov Chain; Non-stationary Process; Regime Switching

Introduction

This condition that a model for a financial time series remains fixed over a very long period rarely happens in practice. Indeed, economic variables may behave differently during a long-run time. This fact makes economists do structural change analysis. Abrupt changes are a fundamental feature of financial data. These shifts may happen in mean or variance or both or in the covariance structure of a financial time series. There are many useful techniques for modeling shifts such as change point analysis, use of dummy variables, threshold analysis and regime switching method, referring to Hamilton (1994).

Usually, the distributional properties of a financial time series relate to its mean and auto-covariance functions (ACVF). The ACVF is an important function for studying the covariance structure of each process. For example, it is useful tool for determining the type of every process. For a review on applications of ACVF, referring to Brockwell and Davis (2002).

As described in above, a useful approach to model a financial time series is to consider different behaviors (structural break) in one subsample (or regime) to another. If the dates of the regimes are known, modeling can be done using dummy variables. However, when these dates are unknown, or they are not fixed, we can consider them as random variables and a regime switching model may be applied.

Regime switching methods are very useful for modeling the movement of price of a specified stock market or formulating the dynamic correlations or pricing volatility swaps under Heston's stochastic volatility model, referring to couch (2009) and Pelletier (2004).

Derivation

In this section, we consider the derivation of ACVF for RS-AR (1) time series. Firstly, we derive a formula for a time varying AR (1) process and finally, then we use this function to derive the ACVF formula for RS-AR (1) process. Therefore, consider a first order zero mean regime switching auto-regressive RS-AR (1) process defined by

$$Y_t = \phi_{S_t^1} Y_{t-1} + \sigma_{S_t^2} Z_t,$$

where

$$|\phi_r| \le |\phi| < 1 \qquad \sigma_r \le M$$

for all integers number r. Here, S_t^1 and S_t^2 are two independent Markov chains with the same state space $\mathfrak{S}$. Let Z_t be a white noise process with zero mean and variance σ^2. In this model, we assume that Z_t is independent of Y_{t-1} and S_t^1 and S_t^2 are independent of Y_{t-1} and Z_t. In this paper, we are going to derive an expression for ACVF in this non-stationary process, i.e.,

$$\gamma_t(h) = \mathrm{cov}(Y_{t+h}, Y_t),\ h = 0, \pm1, \pm2, \dots .$$

The regime switching model deals with the capturing structural changes in the underlying financial time

series using the time ordered observations. An example is Cryer and Chan (2008) who applied this type of time series for modeling the stock returns. In this model, the parameters such as mean and/or volatility vary through the sample; in fact they are functions of some Markov chain processes. This model has been applied in regression analysis, Box-Jenkins time series and as well as in GARCH modeling of financial problems. An excellent reference in this field is Zivot and Wang (2006).

To derive the ACVF, first consider a zero mean AR (1) process with time varying parameters defined by

$$Y_t^* = \phi_t^* Y_{t-1}^* + \sigma_t^* Z_t^*,$$

where

$$|\phi_r^*| \leq |\phi^*| < 1 \text{ and } \sigma_r^* \leq M^*$$

and

$$Z_t^* \sim \mathrm{WN}(0, \sigma^{*2}).$$

Suppose that

$$E(Y_t^*) = 0$$

and note that

$$v_t^* = \mathrm{var}(Y_t^*) \leq \phi^{*2} v_{t-1}^* + M^* \sigma^{*2}.$$

Therefore, we see that

$$\begin{aligned} v_t^* &\leq M^*\sigma^{*2}(1+\phi^{*2}+\ldots+\phi^{*2(t-1)}) + \phi^{*2t}v_1^* \\ &= M^*\sigma^{*2}\frac{(1-\phi^{*2t})}{(1-\phi^{*2})} + \phi^{*2t}v_1^* \\ &\leq M^*\sigma^{*2}\frac{(1-\phi^{*2t})}{(1-\phi^{*2})} + v_1^* = UB. \end{aligned}$$

Here, UB means the upper bound.

Using a recursive solution, we find that

$$\begin{aligned} E(Y_t^* - \sum_{i=0}^{k} d_i^{*t} Z_{t-i}^*)^2 &= \prod_{i=0}^{k} \phi_{t-i}^{*2} E(Y_t^{*2}) \\ &\leq |\phi^*|^{2(k+1)} UB \\ &\to 0, \text{ as } k \to \infty, \end{aligned}$$

where the coefficients are given as follows

$$d_i^{*t} = \sigma_{t-i}^* \prod_{j=1}^{i} \phi_{t-j+1}^*.$$

Therefore, with probability one, we conclude (by letting *k* goes to infinity) that

$$Y_t^* = \sum_{i=0}^{\infty} d_i^{*t} Z_{t-i}^*.$$

There is another way to obtain this equation. Let *B* denote the backward operator. Note that

$$Y_t^* = \frac{1}{1-\phi_t^* B}(\sigma_t^* Z_t^*),$$

which equals to (using Taylor expansion)

$$Y_t^* = \sum_{i=0}^{\infty} \prod_{j=1}^{i} \phi_{t-j+1}^* B^i(\sigma_t^* Z_t^*) = \sum_{i=0}^{\infty} d_i^{*t} Z_{t-i}^*.$$

Using the above equation, it is not difficult to see that

$$\mathrm{cov}(Y_t^*, Y_{t+h}^*) = \sigma^{*2} \sum_{i=0}^{\infty} d_i^{*t} d_{i+h}^{*t}.$$

Now, we are in a position to calculate the ACVF. Define the sigma-field *F* constructed using the whole information of two Markov chains up to time *t* as follows

$$F = \sigma(\{S_{t-i}^1\}_{i=0}^{\infty}, \{S_{t-i}^2\}_{i=0}^{\infty})$$

and let

$$D_i^t = \sigma(S_{t-i}^2) \prod_{j=1}^{i} \phi(S_{t-j+1}^1).$$

It is easy to see that

$$E(Y_t Y_{t+h}|F) = \sigma^2 \sum_{i=0}^{\infty} D_i^t D_{i+h}^t.$$

Therefore, using total probability law, we obtain

$$\gamma_t(h) = \sigma^2 \sum_{i=0}^{\infty} E(D_i^t D_{i+h}^t).$$

However, it seems that calculating the above expectation will be hard in practice. In the following, we employ a technique which simplifies calculation of this mean. To this end, supposing that *U(t)* is a Markov chain with a finite state space. To calculate

$$E(\prod_{j=1}^{p} U_j),$$

we first suppose that *p*=2 then

$$E(U_1 U_2) = E(U_1 E(U_2|U1)).$$

Next, let *p=3*

$$
\begin{aligned}
E(U_1U_2U_3) &= E(U_1U_2E(U_3|U_1,U_2)) \\
&= E(U_1U_2E(U_3|U_2)) : \text{Markov property} \\
&= E(U_1E(U_2|U_1)E(U_3|U_2)).
\end{aligned}
$$

Since the conditional distributions exist, this expectation can be calculated. For other choices of p the same method is applied. This technique is applicable for

$$E(\prod_{j=1}^{p} h_j(U_j))$$

for some measurable functions

$$h_j,\ j = 1, 2, ..., p,$$

for more description on this method, referring to Iacus (2008). Note that, in practice, we can estimate the

$$E(D_i^t D_{i+h}^t)$$

using a Monte Carlo technique including some variance reduction methods, referring to Brazzale et al. (2007).

Application

In this section, we use an application of formulae obtained from the previous section. The ACVF of RS-AR(1) is a function of expectation of some product of Markov chains. As shown in the preceding, this expectation decays exponentially, as *h* goes to infinity.

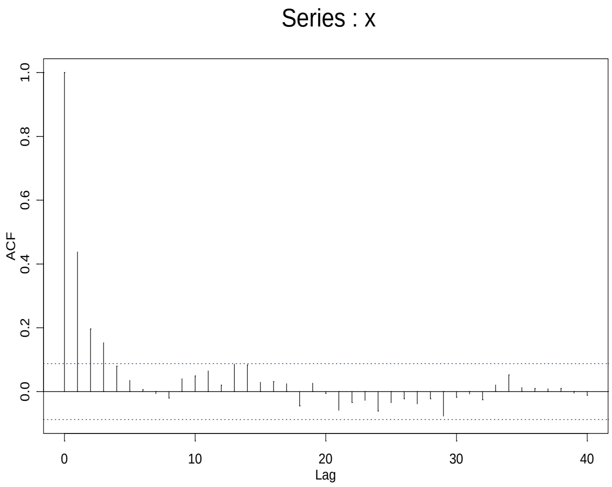
Therefore, this property is transferred to ACVF. Up to this time, the ACVF behaves like the ACVF of a ordinary AR(1). However, the rate of decay to zero is different from time point *t* to *t + 1*.

Thus, a unified strategy for selecting a RS-AR (1) is to inspect the time series plot of the series. If it appears that there are some changes in its level, mean or variance, and if there is an exponential-wise decay to zero with different rates, then RS-AR (1) model may be appropriate. Estimate the related model parameters and do suitable statistical inference.

In the following examples, we see that how, we can understand the existence of RS-AR (1) model from time series and ACF plots.

Example 1. Here, we consider the seasonally adjusted percentages of growth rate of the U.S. quarterly real GNP from the second quarter of 1947 to the first quarter of 1991. By plotting the time series (referring to Potter (1995)) It is seen that most of growth rates are positive. This idea motivates us to use an regime switching model.

Example 2. Here, we use the S-plus software to simulate an AR-RS (1) model, coefficients 0.3 and 0.5 for state 1 and 2, respectively. The elements of transition probabilities matrix are assumed to be 0.5. The following plot shows the ACF of this time series.



It is seen, there is a considerable value for ACF in k=1 and the ACF decays after 1 exponentially. This suggests a first order regime autoregressive switching model.

Conclusion

In this paper, we employ the auto-covariance function of an AR (1) regime switching model. Its behaviour is studied and it is seen that the auto-covariance function is different from zero significantly and it decays suddenly after *h=1*. These properties are used to specify an AR-RS (1) model.

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